2020 Maths Olympiads Division J Resource Kit B

This is the second of four kits in this special 2020 intra-school implementation of the APSMO Maths Olympiads.

At APSMO, we strongly believe that the ultimate goal of school mathematics, is to develop in our students the ability to solve problems. However, the current educational landscape presents a number of challenges for the implementation of problem solving teaching methods that we know students have used with considerable success. These methods rely largely on students having a go, explaining the strategies they used, and then learning from the strategies that were used by their peers.

In order to provide opportunities for such learning when teaching is being delivered remotely, we have selected a few problems from competitions from previous years. For each of these problems, a number of different solution methods are then suggested, so that students can still be exposed to multiple ways of approaching the problem. This leads to a recognition that solving the problem successfully can be achieved by applying logical and mathematical reasoning in a number of different ways.

Examples of how this kit may be used include:

- Introducing new or different solution methods
- Providing diagrams that support a teacher's or student's explanations
- Offering problem-solving homework (within this kit, there is a single page that includes all of the questions)
- Supporting students' own study as a standalone resource

Further questions and solution methods can also be found in the APSMO resource books, available from www.apsmo.edu.au.

We hope that you will find this resource kit useful. Two more kits will become available in the lead-up to 2020 Olympiads 4 and 5.

B.1) Each of *AB* and *BA* represents a two-digit number having the same digits, but in reverse order.

If the difference of the two numbers is 54, and $A + B = 10$, find both numbers, *AB* and *BA*.

B.2) In a class of 26 students, 15 like vanilla ice cream and 16 like chocolate ice cream.

However, 3 do not like either flavour.

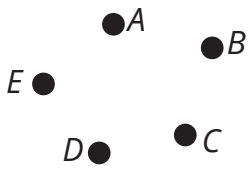
How many students like both vanilla and chocolate ice cream?

B.3) Ming, Natasha, Oliver and Pilar went out to pick blueberries.
Ming picked 17 blueberries.
Natasha picked 15 blueberries.
Oliver and Pilar both picked the same number of blueberries as each other.
After the friends shared the blueberries out equally, each one of them had 20 blueberries.
How many blueberries did Oliver pick?

B.4) The five points shown represent five towns.

Kevin starts at town *A* and visits each of the other towns exactly once.

In how many different orders can Kevin visit the other four towns?



B.5) Each day Jeffrey earns $3 for washing the dishes.

He can earn $5 instead by also sweeping the kitchen.

After ten days, Jeffrey has earned a total of $36.

On how many of these days did Jeffrey sweep the kitchen?

Example Problem B.1

Each of AB and BA represents a two-digit number having the same digits, but in reverse order.

If the difference of the two numbers is 54, and $A + B = 10$, find both numbers, AB and BA.

Strategy 1: Eliminate all but One Possibility

We know that:

$$A + B = 10$$
$$BA - AB = 54$$

Let's try some numbers to get a feel for what this means.

Suppose	A	=	1.
Then, since	$A + B$	=	10,
we can see that	B	=	9.
So,	$BA - AB$	=	$91 - 19$
		=	72.

What does this tell us?

- There aren't many possibilities.
- Since we're finding the difference $BA - AB$, we'll only need to consider values of A and B where B is greater than A.

Trying each possible combination of values for A and B, we have:

A:	1	2	3	4	5
B:	9	8	7	6	5

We can now fill in the values for BA and AB, and find the difference between these values.

A:	1	2	3	4	5
B:	9	8	7	6	5
BA:	91	82	73	64	55
AB:	19	28	37	46	55
$BA - AB$:	$91 - 19 = 72$	$82 - 28 = 54$	$73 - 37 = 36$	$64 - 46 = 18$	$55 - 55 = 0$

Since $82 - 28 = 54$, AB and BA must be **28** and **82**.

Strategy 2: Construct an Expression

Let's consider what it means for AB to represent a two-digit number with the digits A and B.

In our base-10 numbering system, the right-most digit represents the number of ones. Then we have tens, hundreds, thousands, and so on.

Thousands	Hundreds	Tens	Ones

If there's a 2 in the tens place, that represents the value 20.
Having 3 in the tens place represents 30.

So having A in the tens place would actually represent a value that is equal to $10 \times A$.

Thousands	Hundreds	Tens	Ones
		A	

We can see that the two-digit number AB must have the value $10 \times A + B$.

Likewise, the two-digit number BA would have the value $10 \times B + A$.

Thousands	Hundreds	Tens	Ones
		A	B
		B	A

This means that the value $BA - AB$ is really equal to $(10 \times B + A) - (10 \times A + B)$.

To work out the value $BA - AB$:

$$\begin{aligned}&(10 \times B + A) - (10 \times A + B)\\ &= 10 \times B + A - 10 \times A - B\\ &= 9 \times B - 9 \times A\\ &= 9 \times (B - A).\end{aligned}$$

According to the question, $BA - AB = 54$, so we have:

$$\begin{aligned}9 \times (B - A) &= 54\\ B - A &= 54 \div 9\\ &= 6.\end{aligned}$$

We now know that:

$$\begin{aligned}A + B &= 10\\ B - A &= 6\end{aligned}$$

Since they're both equations, we can combine them and the result will still be an equation.

$$\begin{aligned}A + B + B - A &= 10 + 6\\ 2 \times B &= 16\\ B &= 8\\ \text{and so } A &= 2.\end{aligned}$$

Therefore AB and BA must be **28** and **82**.

Answer: 28 and 82

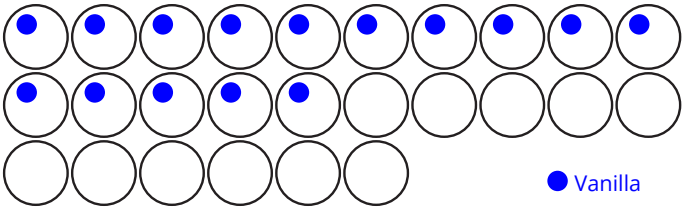
Example Problem B.2

In a class of 26 students, 15 like vanilla ice cream and 16 like chocolate ice cream.
However, 3 do not like either flavour. How many students like both vanilla and chocolate ice cream?

Strategy: Draw a Diagram (1)

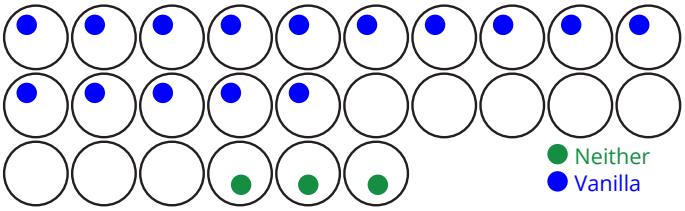
Let's draw our **26** students.

We know that 15 students like vanilla, so let's put a blue dot on them to show that they like vanilla.



There are 16 students who like chocolate, but it's hard to tell which ones to choose because some of them also like vanilla.

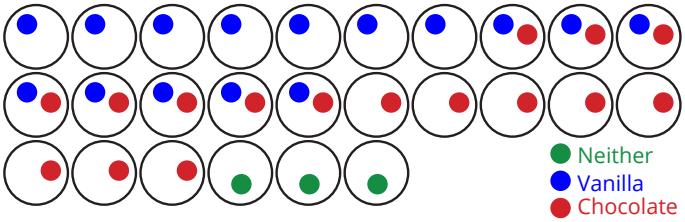
We do know, however, that there are 3 students who do not like either chocolate or vanilla. Let's put a green dot on those three.



The students who like chocolate won't have a green dot.

So they must be all of the remaining (blank) students plus some who also like vanilla.

Let's mark them with a red dot.

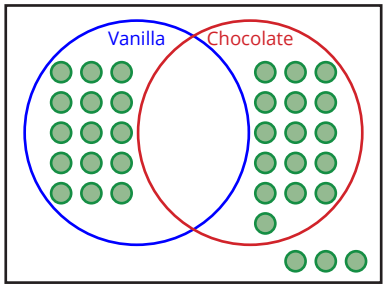


Students who like vanilla have a blue dot. Students who like chocolate have a red dot.
So students who like both vanilla and chocolate ice cream, have both a blue dot and a red dot.
There are **8** students who like both vanilla and chocolate ice cream.

Strategy: Draw a Diagram (2)

In this class, there are:

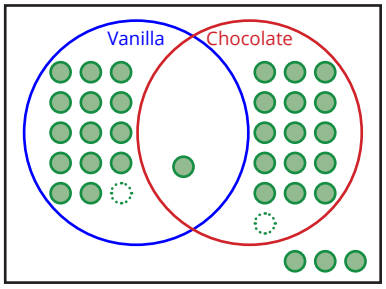
- 15 students who like vanilla,
- 16 students who like chocolate, and
- 3 students who like neither.



That's 15 + 16 + 3 = 34 students, which is too many.

Let's put one student into the overlapping section.

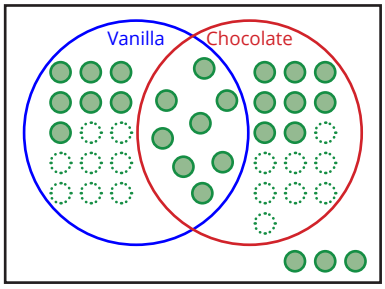
This means they'll be counted for both vanilla and chocolate, so we can take one student out of both of those sections.



Now, there are 14 + 1 + 15 + 3 = 33 students, reducing the total by **1**.

Every time we add a student to the overlapping section, we can take one student out of both vanilla-only and chocolate-only.

So, to reduce the number of students from **34** to **26**, we'll need to do this procedure **34 – 26 = 8** times.



Therefore there are **8** students who like both vanilla and chocolate.

Answer: 8

Example Problem B.3

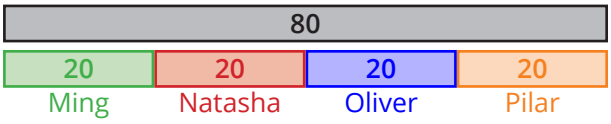
Ming, Natasha, Oliver and Pilar went out to pick blueberries.
Ming picked 17 blueberries.
Natasha picked 15 blueberries.
Oliver and Pilar both picked the same number of blueberries as each other.
After the friends shared the blueberries out equally, each one of them had 20 blueberries.
How many blueberries did Oliver pick?

Strategy 1: Draw a Diagram, and Work Backwards

Let's start at the end of the problem, and work towards the beginning.

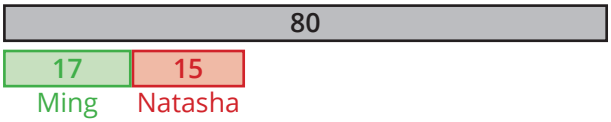
Everyone ended up with **20** blueberries.

Since they ended up with **20** blueberries each, there must have been **4 × 20 = 80** blueberries in total.

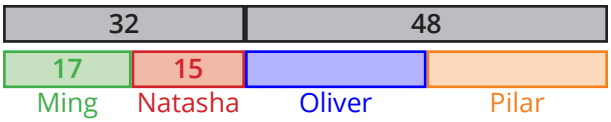


Originally, Ming had picked 17 blueberries.

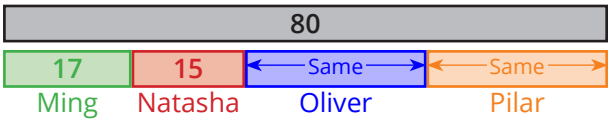
Natasha had picked 15.



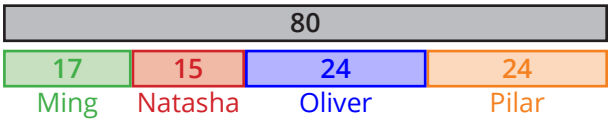
So together, Oliver and Pilar must have picked the other **80 – 17 – 15 = 48** blueberries.



Oliver and Pilar both picked the same number of blueberries as each other.



So Oliver must have picked **48 ÷ 2 = 24** blueberries.



Strategy 2: Build a Table, and Work Backwards

Let's make a table to talk about the blueberry situation after each of these events.

We'll include the information we have been given.

After picking blueberries	After sharing blueberries
Ming: **17** Natasha: **15** Oliver: **?** Pilar: **?**	Ming: **20** Natasha: **20** Oliver: **20** Pilar: **20**

We can think of the friends putting together all of the blueberries they picked, and then sharing them out.

How many blueberries were there in total to share out?

After picking blueberries	After sharing blueberries
Ming: **17** Natasha: **15** Oliver: **?** Pilar: **?**	Ming: **20** Natasha: **20** Oliver: **20** Pilar: **20**
	Total: **4 × 20 = 80**

There were **80** blueberries in total.

Ming picked **17** blueberries.

So Natasha, Oliver and Pilar picked **80 – 17 = 63** blueberries.

Natasha picked **15** blueberries.

So Oliver and Pilar picked **63 – 15 = 48** blueberries.

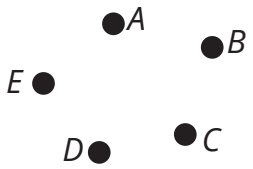
Since Oliver and Pilar both picked the same number of blueberries as each other, then each of them must have picked **48 ÷ 2 = 24** blueberries.

So Oliver picked **24** blueberries.

Answer: 24

Example Problem B.4

The five points shown represent five towns.
Kevin starts at town *A* and visits each of the other towns exactly once.
In how many different orders can Kevin visit the other four towns?



Strategy 1: Make an Organised List

Kevin starts at *A*.
There are **4** towns left.
His second town could be *B*, *C*, *D* or *E*.

Town 1	Town 2
A	B
	C
	D
	E
C	B
	D
	E
D	B
	C
	E
E	B
	C
	D

Suppose the 2nd town was *B*.
His next town could be *C*, *D* or *E*.
Regardless of which town came 2nd, there will be **3** towns left.

Town 1	Town 2	Town 3
A	B	C
		D
		E
	C	B
		D
		E
	D	B
		C
		E
	E	B
		C
		D

Suppose the 2nd town was *B*, and the 3rd town was *C*.
His next town could be *D* or *E*.
We can see that, after any **3** towns, there will be **2** towns left.

Town 1	Town 2	Town 3	Town 4
A	B	C	D
			E
		D	C
			E
		E	C
			D
	C	B	D
			E
		D	B
			E
		E	C
			D
	D	B	C
			E
		C	B
			E
		E	B
			C
	E	B	C
			D
		C	B
			D
		D	B
			C

Suppose the 2nd town was *B*, the 3rd town was *C*, and the 4th town was *D*.
His next town must be *E*.
After any **4** towns, there will be no choice of where to go. There is only **1** town left.

Town 1	Town 2	Town 3	Town 4	Town 5	No.
A	B	C	D	E	1
			E	D	2
		D	C	E	3
			E	C	4
		E	C	D	5
			D	C	6
	C	B	D	E	7
			E	D	8
		D	B	E	9
			E	B	10
		E	C	D	11
			D	C	12
	D	B	C	E	13
			E	C	14
		C	B	E	15
			E	B	16
		E	B	C	17
			C	B	18
	E	B	C	D	19
			D	C	20
		C	B	D	21
			D	B	22
		D	B	C	23
			C	B	24

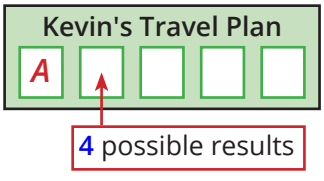
By counting in an organised way, we can be sure we have found every possible order. Let's count them up.
Kevin can visit the other **4** towns in **24** different ways.

Strategy 2: Act with Concrete Materials

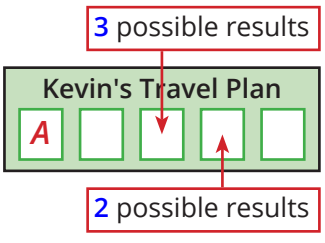
Let's suppose that Kevin has five cards. They are labelled *A*, *B*, *C*, *D* and *E*.

He wants to choose the order of towns by randomly picking cards.
So he turns the cards over and selects the first one, which the question tells us is an *A*.

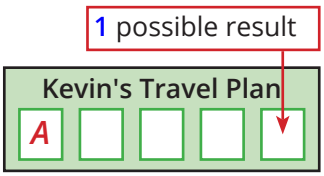
Kevin writes "*A*" in the 1st spot on his travel plan.



He has **4** cards left to choose from.
So there are **4** possible results for the 2nd spot.



He has **3** cards left to choose from for the 3rd spot, and **2** cards left for the 4th spot.



Now there's only **1** card left. That town must be written in the last spot.

So there are
4 × 3 × 2 × 1 = 24 ways
to visit 4 towns after ***A***.

Answer: 24

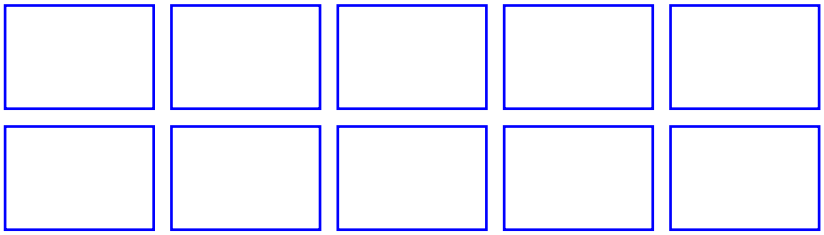
Example Problem B.5

Each day Jeffrey earns $3 for washing the dishes.
He can earn $5 instead by also sweeping the kitchen.
After ten days, Jeffrey has earned a total of $36.
On how many of these days did Jeffrey sweep the kitchen?

Strategy 1: Draw a Diagram

Let's pretend that Jeffrey collects his pay in an envelope.

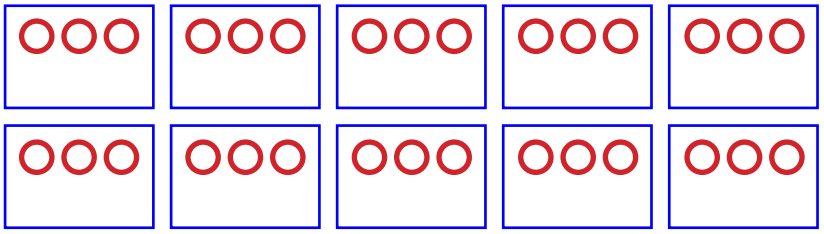
After 10 days, he will have 10 envelopes.



Each day, Jeffrey earns either $3 or $5.

So every day he earns at least $3.

Let's put $3 in each envelope.



So far, that's 10 × $3 = $30.

Jeffrey actually earned **$36**, so some of the envelopes must have had **$5** in them.

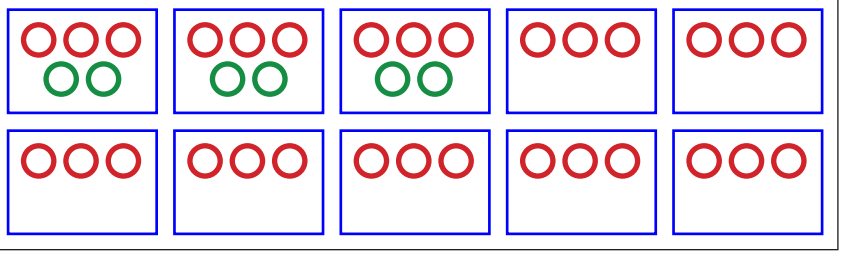
Let's turn some of the envelopes into **$5** envelopes.

Changing one envelope makes it **$32**.

Changing a second one makes it **$34**.

Changing a third makes it **$36**.

This matches the question.



Since Jeffrey earned **$5** on **3** of the days, he must have swept the kitchen on **3** days.

Strategy 2: Use Algebra

Let x represent the number of days that Jeffrey swept the kitchen.

He was paid **$5** for each of those days.
In total, that's **$(5 × x)**.

There were also **10 – x** days on which Jeffrey only washed dishes.

He was paid **$3** for each of those days.
In total, that's **$(3 × (10 – x))**.

After ten days, Jeffrey has earned a total of **$36**.

Putting that all together, we have:

$$\begin{aligned} (5 \times x) + (3 \times (10 - x)) &= 36 \\ 5x + 30 - 3x &= 36 \\ 2x &= 36 - 30 \\ &= 6 \\ x &= 3 \end{aligned}$$

Since x represents the number of days that Jeffrey swept the kitchen, we can see that Jeffrey swept the kitchen on **3** days.

Answer: 3

Answers

B.1) Each of *AB* and *BA* represents a two-digit number having the same digits, but in reverse order.
If the difference of the two numbers is 54, and $A + B = 10$, find both numbers, *AB* and *BA*.

Answer: 28 and 82

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However, 3 do not like either flavour.
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Answer: 8

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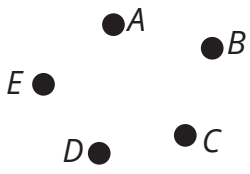

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B.5) Each day Jeffrey earns $3 for washing the dishes.
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On how many of these days did Jeffrey sweep the kitchen?

Answer: 3