2020 Maths Olympiads Division J Resource Kit A

MATHS OLYMPIAD

Welcome to this special 2020 intra-school implementation of the APSMO Maths Olympiads.

At APSMO, we strongly believe that the ultimate goal of school mathematics, at all times, is to develop in our students the ability to solve problems. However, the current educational landscape presents a number of challenges for the implementation of problem solving teaching methods that we have, to date, seen used with considerable success - methods that rely largely on students having a go, explaining the strategies they used, and then learning from the strategies that were used by their peers.

In order to provide opportunities for such learning when teaching is being delivered remotely, we have selected a few problems from competitions from previous years. For each of these problems, a number of different solution methods are then suggested, so that students can still be exposed to multiple ways of approaching the problem. This leads to a recognition that a successful resolution can be achieved by applying logical and mathematical reasoning in a number of different ways.

Examples of how this kit may be used include:

- Introducing new or different solution methods
- Providing diagrams that support a teacher's or student's explanations
- Offering problem-solving homework (within this kit, there is a single page that includes all of the questions)
- Supporting students' own study as a standalone resource

and so on.

Further questions and solution methods can also be found in the APSMO resource books, available from www.apsmo.edu.au.

We hope that you will find this initial resource kit useful. Three more kits will become available in the lead-up to 2020 Olympiads 3, 4 and 5.

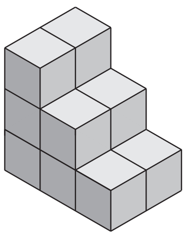
2020 Maths Olympiads Division J Resource Kit A

1.1) Sara said, "If you divide my age by 3 and then add 8 years, the result is my age."
How old is Sara, in years?

1.2) The sum of 3 consecutive numbers is 15 more than the greatest of them.
What is the greatest of the three numbers?

1.3) Joseph put candles on a cake.
He lit $\frac{1}{3}$ of those candles.
Then he lit 5 more of the candles on the cake.
As a result, exactly $\frac{1}{2}$ of the candles on the cake were lit.
What was the total number of candles on the cake?

1.4) Twelve cubes are glued together to form the object shown.
The length of an edge of each cube is 3 cm.
The entire object is dipped in and out of a can of paint.
How many square centimetres are covered in paint?



1.5) A shopkeeper sells house numbers.
She has a large supply of the digits 4, 7, and 8, but no other digits.
How many different three-digit house numbers could be made using only the digits in her supply?

Example Problem 1.1

Sara said, "If you divide my age by 3 and then add 8 years, the result is my age." How old is Sara, in years?

Strategy 1: Guess, Check and Refine

Sara's age must be divisible by **3**.

Let's guess that Sara is **9** years old.

- Dividing Sara's age by **3**: **9** ÷ **3** = **3** years
- Adding **8** years to the result: **3** + **8** = **11** years.

The calculation doesn't work if Sara is **9** years old.

The result differs from her age by **11** - **9** = **2** years.

Let's see what happens if Sara is **6** years old.

- Dividing Sara's age by **3**: **6** ÷ **3** = **2** years
- Adding **8** years to the result: **2** + **8** = **10** years.

When we guess that Sara is **6** years old, the result differs from her age by **10** - **6** = **4** years.

As our guesses get smaller, the differences get larger.
Perhaps if we guess a larger number for Sara's age, the difference will get smaller.

Let's guess that Sara is **12** years old.

- Dividing Sara's age by **3**: **12** ÷ **3** = **4** years
- Adding **8** years to the result: **4** + **8** = **12** years.

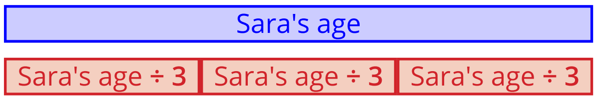
The result is the same as our guess.

Therefore Sara is **12** years old.

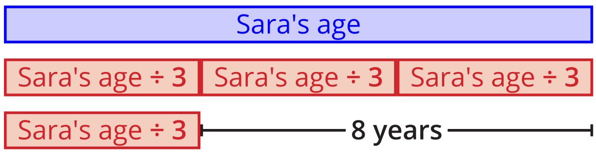
Strategy 2: Draw a Diagram, and Work Backwards

Let's use a bar to represent Sara's age.

We can then use bars to represent dividing Sara's age by 3.

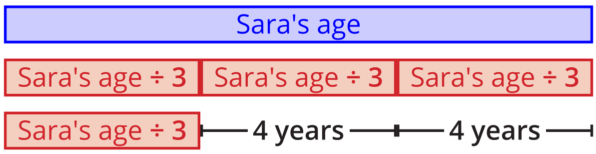


If we add **8** years to Sara's age ÷ **3**, the result will be Sara's age.

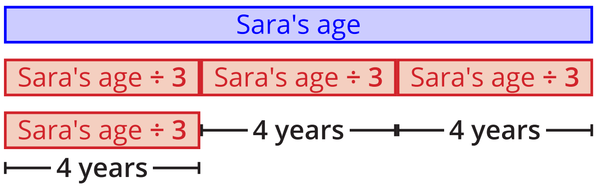


From our diagram, we can see that
2 × Sara's age ÷ **3** = **8** years.

Therefore Sara's age ÷ **3** = **8** ÷ **2** = **4** years.



Therefore Sara's age = **4** + **4** + **4** = **12 years.**



Strategy 3: Use Algebra

Let x represent Sara's age.

Then we can set up the following equation: $\frac{x}{3} + 8 = x$

Multiply both sides by 3: $x + 24 = 3x$

Subtract x from both sides: $24 = 2x$

Divide both sides by 2: $12 = x$

Since x represents Sara's age, we have determined that Sara must be **12** years old.

Answer: 12

Example Problem 1.2

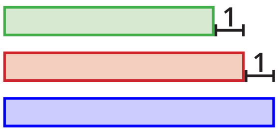
The sum of 3 consecutive numbers is 15 more than the greatest of them.

What is the greatest of the three numbers?

Strategy 1: Draw a Diagram

There are **3** consecutive numbers.

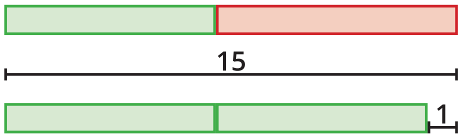
So each number is **1** more than the previous number.



The sum of all **3** numbers is **15** more than the greatest of the **3** numbers.

This means that the **2** smallest numbers add up to **15**.

We can replace the middle value with the smallest value plus **1**.



So two of the smallest value must be **15 – 1 = 14**.

The smallest value must be **14 ÷ 2 = 7**.

Therefore, the **3** consecutive numbers are **7**, **8**, **9**.

The greatest of the **3** numbers is **9**.

Strategy 2: Use Algebra

Let **M** represent the smallest number. The **3** consecutive numbers are M, $M + 1$, $M + 2$.

The sum of the **3** numbers is **15** more than the greatest of them.	$M + (M + 1) + (M + 2) = 15 + (M + 2)$
Subtract $(M + 2)$ from each side of the equation.	$M + (M + 1) = 15$
Replace $M + M$ with $2 \times M$.	$2 \times M + 1 = 15$
Subtract **1** from each side.	$2 \times M = 14$
Divide each side by **2**.	$M = 7$

So the greatest of the **3** numbers is **7 + 2 = 9**.

Strategy 3: Build a Table and Find a Pattern

Let's guess that the **3** numbers are **1**, **2**, **3**.

What if the **3** numbers are **2**, **3**, **4**,

or **3**, **4**, **5**?

Can you see a pattern?
Every time we increase the first number by **1**, the difference gets smaller by **2**.
Why might this happen?

It looks like the difference will be **0** (i.e. the two values are the same) when the **3** consecutive numbers are **7**, **8**, **9**.

The 3 numbers			Sum of numbers	3rd number plus 15	Difference
1	2	3	1 + 2 + 3 = 6	3 + 15 = 18	18 – 6 = 12
2	3	4	2 + 3 + 4 = 9	4 + 15 = 19	19 – 9 = 10
3	4	5	3 + 4 + 5 = 12	5 + 15 = 20	20 – 12 = 8
4					6?
5					4?
6					2?
7	8	9			0?

Let's check: The sum of the numbers is **7 + 8 + 9 = 24**. The 3rd number plus 15 is **9 + 15 = 24**.
The two values are equal.

So the greatest of the **3** numbers is **9**.

Answer: 9

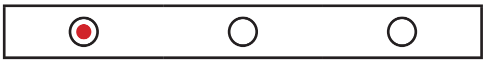
Example Problem 1.3

Joseph put candles on a cake. He lit $\frac{1}{3}$ of those candles. Then he lit 5 more of the candles on the cake. As a result, exactly $\frac{1}{2}$ of the candles on the cake were lit. What was the total number of candles on the cake?

Strategy 1: Guess, Check and Draw a Diagram

Let's guess that there were **3** candles.

Joseph started by lighting just one candle.



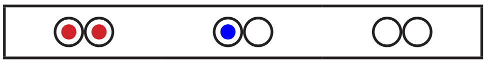
He then lit some more candles, so that exactly half of the candles were lit.

But it's not possible to light exactly half of 3 candles.

Let's guess that there were **6** candles.

Joseph started by lighting 6 ÷ 3 = 2 candles.

He then lit some more candles, so that exactly half of the candles were lit.

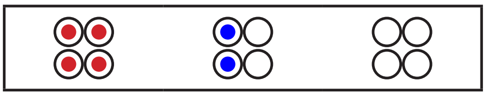


Joseph lit **5** more candles to end up with half of the candles being lit.

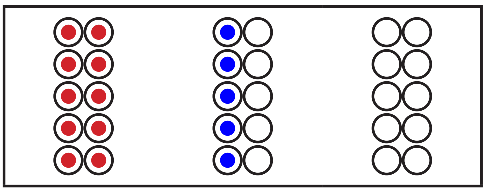
We only lit **1** more candle to end up with half of the candles being lit.

Can we use our diagram to work out how many candles Joseph had?

If we doubled the number of candles, we would light **2** more candles to end up with half of the candles being lit.



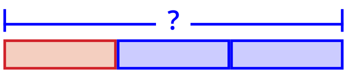
So if we multiplied the number of candles by **5**, we would need to light **5** more candles to end up with half of the candles being lit.



According to the diagram, Joseph must have had **30** candles.

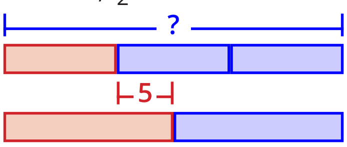
Strategy 2: Draw a Diagram

We want to find out the total number of candles.

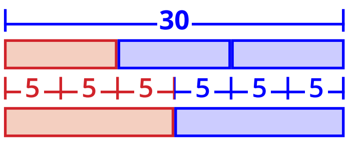


To begin with, $\frac{1}{3}$ of them were lit.

After lighting 5 more candles, $\frac{1}{2}$ of them were lit.



Just visually, it seems that the 5 extra candles represent half of one-third of the total number.



If so, there would be **6 × 5 = 30** candles in total.

Let's check: Joseph lit $\frac{1}{3}$ × **30** = **10** candles. Then he lit **5** more for a total of **10** + **5** = **15** candles.

As a result, exactly $\frac{1}{2}$ of the candles on the cake were lit.

Strategy 3: Perform the Calculation

Let x represent the number of candles.

Joseph started by lighting $\frac{1}{3}x$ candles.

He then lit 5 more candles.

So far, Joseph has lit $\frac{1}{3}x + 5$ candles.

As a result, exactly $\frac{1}{2}x$ candles are lit.

Now we can set up the following equation:

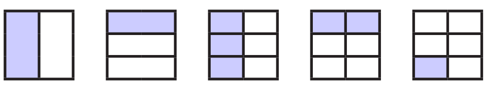
$$\frac{1}{3}x + 5 = \frac{1}{2}x$$

If we subtract $\frac{1}{3}x$ from both sides of this equation, we have:

$$5 = \frac{1}{2}x - \frac{1}{3}x$$

$$= (\frac{1}{2} - \frac{1}{3})x.$$

To determine the value of $(\frac{1}{2} - \frac{1}{3})$:



$$\frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$$

Therefore:

$$5 = \frac{1}{6}x$$

Multiplying both sides by 6, we have

$$30 = x$$

Since x represents the number of candles, we can see that there were **30** candles in total.

Answer: 30

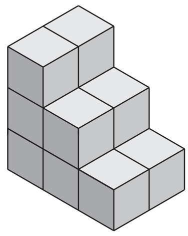
Example Problem 1.4

Twelve cubes are glued together to form the object shown.

The length of an edge of each cube is 3 cm.

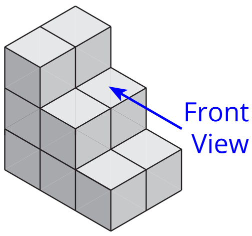
The entire object is dipped in and out of a can of paint.

How many square centimetres are covered in paint?

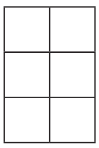


Strategy 1: Draw a Diagram

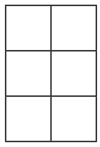
Let's define this direction as the "front view" of the staircase.

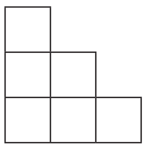


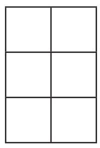
Looking squarely from the front, the staircase would look like this.

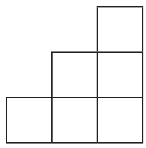


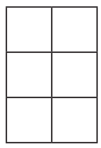
Using this definition, we can draw the six different views of this object.

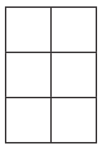

Top View


Left View

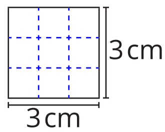

Front View


Right View


Back View


Bottom View

Each cube face has an area of **3 cm × 3 cm = 9 cm²**.



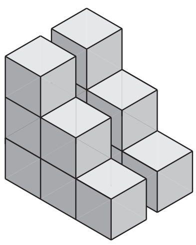
Since six cube faces are visible from each direction, the area of each view is **6 × 9 cm² = 54 cm²**.

Therefore **6 × 54 = 324** square centimetres are covered in paint.

Strategy 2: Divide a Complex Shape, and Use Symmetry

The staircase is two blocks wide.

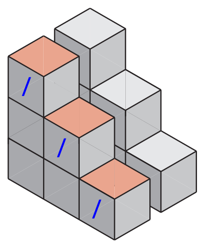
If we split it in half, we will have two sections, each one block wide, that are mirror images of each other.



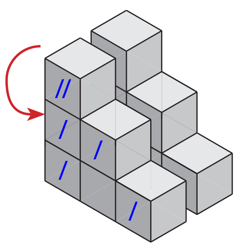
Let's paint the half-staircase on the left hand side.

First we'll consider each face that points upwards.

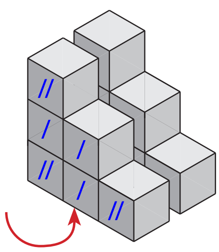
If a face would get painted, we can mark that cube with a tally mark.



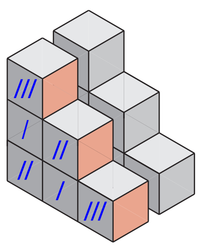
Next let's paint the faces on the back of the staircase,



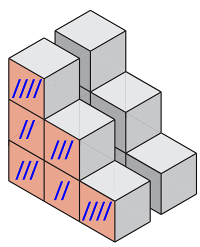
the under-side of the staircase,



the front face (or stair risers),



and the side of the staircase.



There are **4 + 2 + 3 + 3 + 2 + 4 = 18** painted faces on the left half of the staircase.

By symmetry, there would be just as many painted faces on the right half of the staircase.

We have counted **18 + 18 = 36** painted faces.

Each painted face has an area of **3 cm × 3 cm = 9 cm²**.

Therefore the number of painted square centimetres is **36 × 9 = 324** square centimetres.

Answer: 324 (square centimetres)

Example Problem 1.5

A shopkeeper sells house numbers.

She has a large supply of the digits 4, 7, and 8, but no other digits.

How many different three-digit house numbers could be made using only the digits in her supply?

Strategy 1: Make an Organised List

Let's start with the smallest possible three-digit number using these digits.

What might this number be?

If 444 is the smallest, then what would the next smallest numbers be?

444, 447, 448

Counting up from 448, we can't make 449, or any numbers in the **450**s or **460**s.

So our next numbers will be in the 470s and 480s.

444, 447, 448
474, 477, 478
484, 487, 488

That gives us all of the numbers in the 400s.

There will also be similar sets of numbers in the 700s and 800s.

444, 447, 448	744, 747, 748	844, 847, 848
474, 477, 478	774, 777, 778	874, 877, 878
484, 487, 488	784, 787, 788	884, 887, 888

By counting in an organised way, we can be sure that we have found all of the possible numbers.

We found **three** sets of **9** numbers.

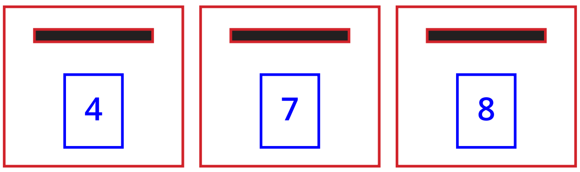
So there are **3 × 9 = 27** different three-digit numbers.

Strategy 2: Act with Concrete Materials

Let's pretend that we have a letterbox.

We'll start with a 1-digit house number.

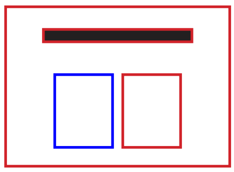
Since we can choose a digit in one of 3 ways, it is possible to have 3 different house numbers.



What if it was a 2-digit house number?

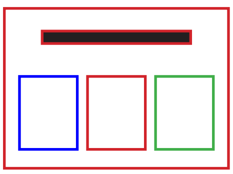
For every possible first digit, there are 3 possible 2nd digits.

This gives us 3 × 3 = 9 different house numbers.



4 → 44, 47, 48
7 → 74, 77, 78
8 → 84, 87, 88

What if it was a 3-digit house number?



For every possible combination for the first 2 digits, there are 3 possible 3rd digits.

44 → 444, 447, 448
47 → 474, 477, 478
48 → 484, 487, 488
74 → 744, 747, 748
77 → 774, 777, 778
78 → 784, 787, 788
84 → 844, 847, 848
87 → 874, 877, 878
88 → 884, 887, 888

There are now 3 × 3 × 3 = 27 different combinations.

So **27** different three-digit house numbers can be made using the digits 4, 7 and 8.

Answer: 27

Answers

1.1) Sara said, "If you divide my age by 3 and then add 8 years, the result is my age."
How old is Sara, in years?

Answer: 12

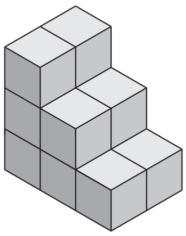
1.2) The sum of 3 consecutive numbers is 15 more than the greatest of them.
What is the greatest of the three numbers?

Answer: 9

1.3) Joseph put candles on a cake. He lit $\frac{1}{3}$ of those candles.
Then he lit 5 more of the candles on the cake.
As a result, exactly $\frac{1}{2}$ of the candles on the cake were lit.
What was the total number of candles on the cake?

Answer: 30

1.4) Twelve cubes are glued together to form the object shown.
The length of an edge of each cube is 3 cm.
The entire object is dipped in and out of a can of paint.
How many square centimetres are covered in paint?



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1.5) A shopkeeper sells house numbers.
She has a large supply of the digits 4, 7, and 8, but no other digits.
How many different three-digit house numbers could be made using only the digits in her supply?

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